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APPROXIMATE METHOD FOR THE SOLUTION TO THE EQUATIONS FOR PARALLEL AND MIXED-FLOW MULTI-CHANNEL HEAT EXCHANGERS*

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Abstract—Approximate solutions for the steady state lumped linear formulation of heat transfer in multi-channel parallel-and-mixed-flow heat exchangers are obtained using polynomial approximations for the temperatures in each stream. Simple second and third degree polynomials result in acceptable errors when compared with the exact solutions. The method is especially suited for the practical treatment of exchangers with a large number of channels when other methods cease to be applicable. Extension of the method to treat the non-linear and transient problem is outlined.

NOMENCLATURE

a_{ik}, specific heat transference from channel i to channel k;

$\bar{a}_{ik}$, dimensionless heat transfer conductance (number of transfer units of channel i to k)

$$\bar{a}_{ik} = \frac{U_{ik}F_{ik}}{C_i} = N_{TUik};$$

c, specific heat of fluid;

c_{ik}, coefficient of the polynomial for t_i;

C, fluid capacity rate (water equivalent), $C = Wc$;

F_{ik}, area of heat exchange between streams i and k, $F_{ik} = Lp_{ik}$;

F_{ci}, cross section area of channel i;

h, convective heat transfer coefficient;

L, length of channel in x-direction;

m, order of polynomial;

n, number of channels;

N_{TU}, number of transfer units;

p_{ik}, common perimeter of channels i and k;

R_w, resistance of the wall;

t_i, temperature of fluid in channel i;

$\bar{t}$, approximate solution;

$\bar{t}^{(m)}$, approximate solution using polynomials of order m;

$\hat{t}$, limiting approximate solution;

$\bar{t}_i^j$, value of $\bar{t}_i$ at ϕ_j;

u_i, velocity of fluid in channel i, $u_i = W_i/F_{ci}$;

u, ratio of capacity rates, $u = C_h/C_c$;

U_{ik}, overall conductance for heat transfer between channels i and k;

U_e, effective heat transfer conductance defined by equation (11);

W, flow rate of fluid.

Greek symbols

ε, effectiveness of the exchanger;

ϕ, dimensionless space coordinate, $\phi = x/L$;

ϕ_j, nodal points for approximating function;

τ, time;

$\bar{\tau}$, dimensionless time, $\bar{\tau} = \tau \dfrac{u_i}{L}$.

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Subscripts
- i, channel;
- b, boundary;
- c, cold;
- h, hot;
- t, total.

INTRODUCTION

THE STEADY state lumped formulation of heat transfer in multi-channel parallel-and-mixed-flow heat exchangers leads to a system of ordinary differential equations of the first order. The number of equations obtained is equal to the number of fluid streams. The equations are linear, provided the thermophysical properties of fluid along each stream can be assumed constant; otherwise a nonlinear system of equations results.

Wolf [1], in a review and extension of the general theory for the linear problem demonstrates a procedure for obtaining exact solutions. Unfortunately, there is one omission in his assumptions, which makes the proposed procedure not applicable to the general problem in some cases. These cases are discussed elsewhere [2]. Furthermore, even if the results of his work were applicable, the solution would be extremely laborious, since in practical applications, i.e. plate exchangers, the number of channels, n, is usually large ($n = 20$–40) and the method thus requires in the first instance finding all eigenvalues and eigenvectors of a matrix of the order n. Exact solutions for $n < 10$ are given for some cases by Mennicke [3, 4], and thus for $n > 10$, the use of a numerical method would seem in order.

Numerical solution of the problem using finite difference methods allows formulation as either an initial or boundary problem.

Standard numerical techniques such as Runge–Kutta or predictor-corrector methods have been used [5, 6]. They require formulation as a pure initial value problem and since only t_i^0 and t_4^0 are known (Fig. 1) for a typical multi-pass exchanger, difficulties soon arise since all initial conditions associated with the 'return stream' must be guessed. Foote [6] solves the problem by the shooting method; numerically a very time consuming method. Apparently, there has not been an attempt to solve the problem as a boundary value problem as is done here.

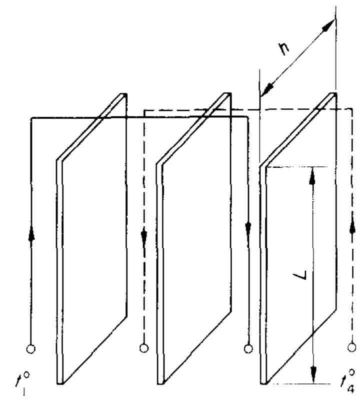

FIG. 1. Typical exchanger geometry.

In this paper, a general numerical method is presented. The method is, in essence, an integral method using polynomial approximations similar to those used in heat conduction problems [7]. The method is applicable to both the linear and nonlinear problem with any boundary condition and allows nonuniform geometry along the stream. It can also be extended to consider the transient behaviour of the exchanger. Numerical results obtained by the presented method are compared with the exact solutions as given by Mennicke.

In the formulation of this problem there are two possibilities: to formulate the problem for temperatures, or for temperature differences. This latter approach reduces the number of equations by one; however, the method based on temperature is utilized here as the simplification for large n is immaterial and moreover, the use of the difference method is impossible for nonlinear problems.

FORMULATION OF THE LINEAR PROBLEM AND THE METHOD OF SOLUTION

The system of equations for temperatures t_i, with x the common space variable along all channels, is [1]

$$\frac{dt_i}{dx} + \sum_{k=1}^{n} a_{ik}(t_i - t_k) = 0 \quad \begin{matrix} i = 1, 2, \ldots, n \\ x \in (0, L) \end{matrix} \tag{1}$$

where

$$a_{ik} = \frac{U_{ik}\, p_{ik}}{C_k}$$

is the specific heat transference (N_{TU}) between channels i and k [8]. Assuming a_{ik} constant, system (1) is linear. In what follows, it is advantageous to write the system of equations (1) in dimensionless form with respect to $x(\phi = x/L)$ that is

$$\frac{dt_i}{d\phi} + \sum_{k=1}^{n} \bar{a}_{ik}(t_i - t_k) = 0, i = 1, 2, \ldots, n \tag{2}$$

where

$$\bar{a}_{ik} = a_{ik} \,.\, L = \frac{U_{ik}\, F_{ik}}{C_i},$$

where F_{ik} is the heat transfer area between streams i and k ($\bar{a}_{ik}$ is the number of heat transfer units for channel i). In the event x is a curvelinear coordinate, L will be different for each stream. As the scaling of temperature is independent, t may be considered dimensionless or not.

Following the idea of the integral method, an approximate solution to t_i is assumed to be a polynomial of degree m,

$$\bar{t}_i = \sum_{k=0}^{m} c_{ik}\, \phi^k. \tag{3}$$

This polynomial is determined by $m + 1$ values of $\bar{t}_i$:

$$\begin{aligned} \bar{t}_i^0 &= \bar{t}_i(\phi_0) \\ \bar{t}_i^1 &= \bar{t}_i(\phi_1) \\ &\vdots \\ \bar{t}_i^m &= \bar{t}_i(\phi_m). \end{aligned}$$

It is now possible to assume $0 = \phi_0 < \phi_1 < \ldots < \phi_m = 1$, which is especially advantageous for the treatment of the boundary conditions. The coefficient for (3) will be a linear combination of $\bar{t}_i^0, \ldots, t + 1_i^m$. Substitution of function (3) in the ith equation of (2) and integrating between ϕ_v and ϕ_u results in

$$\int_{\phi_v}^{\phi_u} \frac{dt_i}{d\phi} d\phi + \sum_k \int_{\phi_v}^{\phi_u} a_{ik}[\bar{t}_i - \bar{t}_k]\, d\phi = 0$$

or

$$\bar{t}_i(\phi_u) - \bar{t}_i(\phi_v) = -\sum_k \Big(\int_{\phi_v}^{\phi_u} \bar{a}_{ik}[\bar{t}_i - \bar{t}_k]\, d\phi\Big).$$

Upon integrating, $\bar{t}_i$ and $\bar{t}_k$ are linear functions of $\bar{t}_i^0, \ldots, \bar{t}_i^m$ and $\bar{t}_k^0, \ldots, \bar{t}_k^m$ only. In this way, a linear algebraic equation for temperatures is obtained. Clearly, $(m + 1) \times n$ such equations are needed to determine all the temperatures $\bar{t}_i^j : j = 0, \ldots, m; i = 1, \ldots, n$. It is useful to choose these equations in the following manner. Choose as the integration limits of each equation m different combinations of the chosen values $\phi_0, \phi_1, \ldots, \phi_m$. Denoting these limits j_k, l_k, the equations will then be of the form,

$$\bar{t}_i^{j_k} - \bar{t}_i^{l_k} = -\sum_k \bar{a}_{ik}\left[\int_{\phi_{l_k}}^{\phi_{j_k}} \bar{t}_i\, d\phi - \int_{\phi_{l_k}}^{\phi_{j_k}} \bar{t}_k\, d\phi\right] \quad \begin{matrix} k = 1, \ldots, m \\ i = 1, \ldots, n. \end{matrix} \tag{4}$$

Every equation, obtained by integrating between other values, ϕ_j, ϕ_l, of the above system of nodal points, will be a linear combination of the above equations and therefore will not yield a new equation since an integral with the limits j and l can always be expressed as the sum of integrals with limits l_k and j_k, as can easily be verified [9]. This method then allows independent choice of integration limits not necessarily coinciding with the nodal points chosen:— for example, from Fig. 3 integration may be preformed from 0 to $\frac{2}{3}$ and from $\frac{1}{3}$ to 1, a process which does not incorporate the nodal point $\frac{1}{2}$ in the limits of

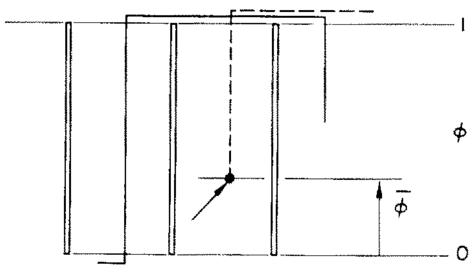

FIG. 2. Non standard entrance to heat exchanger.

integration. This allows then detailed and more accurate examination of specific regions by placing more integration regions in that area. In finite difference methods this is not possible unless an irregular grid is chosen beforehand.

Equations (4) yield $m \times n$ linear algebraic equations. The missing n equations are now just the equations expressing the boundary conditions, and will usually be of the form

$\bar{t}_i(\phi_b) = \bar{t}_b$ —given temperature

or

$\bar{t}_i(\phi_b) = \bar{t}_k(\phi_b)$—continuity of streams k and j where the subscript b refers to the boundary.

The present method will also allow treatment of such cases as when one stream is partially isolated, or external sources or sinks exist, etc. For instance, for the case shown in Fig. 2, by choosing one of the nodal points ϕ_i in the distance $\bar{\phi}$ and by omitting the corresponding integrations between 0 and $\bar{\phi}$, whenever they occur for stream i, the number of equations and unknowns will be reduced and a general solution obtained.

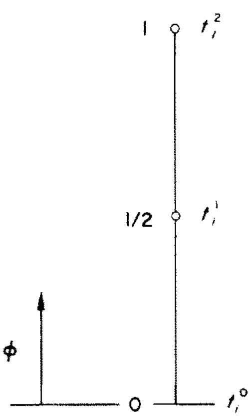

FIG. 3. Temperatures for second degree polynomial.

Generally, a system of $(m + 1) \times n$ linear algebraic equations is obtained which yields, with values $\bar{t}_i^0, \ldots, \bar{t}_i^m$ and the approximations to the temperature distributions (3), the general solutions. It can easily be shown that:

If for $m \to \infty$, such that $\max |\phi_i - \phi_j| \to 0$, the approximating functions $\bar{t}_i^{(m)}$ converge to $\hat{t}_i$, which is continuous and differentiable, then $\hat{t}_i = t_i$, that is the approximate solutions con-

Suppose that $\hat{t}_i(\phi) = \lim_{m \to \infty} \bar{t}_i^{(m)}(\phi)$, then it can be assumed that ϕ_v, ϕ_u are common grid points to all subdivisions, starting from a certain m. Consider

$$\lim_{m \to \infty} \left\{ \int_{\phi_v}^{\phi_u} \sum_k (\bar{t}_i^{(m)} - \bar{t}_k^{(m)})\, \bar{a}_{ik}\, d\phi + \bar{t}_i^{(m)u} - \bar{t}_i^{(m)v} \right\} = 0.$$

Under the assumptions made, this is

$$\int_{\phi_v}^{\phi_u} \sum_k (\hat{t}_i - \hat{t}_k)\, \bar{a}_{ik} + \hat{t}_i(\phi_u) - \hat{t}_i(\phi_v) = 0.$$

Dividing by $h = \phi_u - \phi_v$ and taking the limit

$$\lim_{h \to \infty} \left\{ \frac{1}{h} \int_{\phi_v}^{\phi_u} \sum_k (\hat{t}_i - \hat{t}_k)\, \bar{a}_{ik}\, d\phi \right.$$

$$\left. = -\lim_{h \to 0} \frac{\hat{t}_i(\phi_u) - \hat{t}_i(\phi_v)}{h} \right\}$$

Subsequently the application of the mean value theorem and properties of integral results in

$$\sum_k \bar{a}_{ik}(\hat{t}_i - \hat{t}_k) = -\frac{d\hat{t}_i}{d\phi},$$

which is identical with equation (2).

APPLICATION: POLYNOMIALS OF SECOND AND THIRD ORDER

For the second order polynomial approximation choose $\phi_o = 0$, $\phi_1 = \frac{1}{2}$, $\phi_2 = 1$, according to Fig. 3, where now the dash above t is omitted for clarity. Expressing the polynomial in terms of t_i^0, t_i^1, t_i^2 by any of the known methods, results in

$$t_i = 2\phi^2(t_i^2 - 2t_i^1 + t_i^0) - \phi(t_i^2 - 4t_i^1 + 3t_i^0) + t_i^0.$$

Integrating between 0 and 1

$$\sum_k \bar{a}_{ik} \tfrac{1}{6} [(t_i^2 + 4t_i^1 + t_i^0) - (t_k^2 + 4t_k^1 + t_k^0)] + t_i^2 - t_i^0 = 0 \quad i = 1, \ldots, n \qquad (5)$$

where the left hand side of the equation is simply Simpson's rule for integration across 3 points.

Integrating between 0 and $\frac{1}{2}$ results in

$$\sum_k \bar{a}_{ik} \tfrac{1}{24} [(5t_i^0 + 8t_i^1 - t_i^2) - (5t_k^0 + 8t_k^1 - t_k^2)] + t_i^1 - t_i^0 = 0 \quad i = 1, \ldots, n \qquad (6)$$

and a more symmetric set of equations can be obtained by replacing the first integral by an integral with limits $\frac{1}{2}$ and 1:

$$\sum_k \bar{a}_{ik} \tfrac{1}{24} [(-t_i^0 + 8t_i^1 + 5t_i^2) - (-t_k^0 + 8t_k^1 + 5t_k^2)] + t_i^2 - t_i^1 = 0 \quad i = 1, \ldots, n. \qquad (7)$$

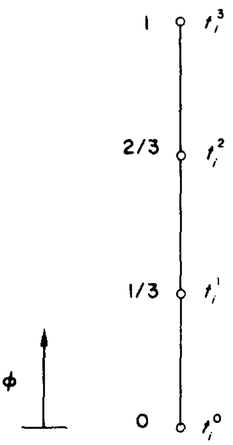

FIG. 4. Temperatures for third degree polynomial.

For polynomials of the third degree, choose $\phi_0 = 0, \phi_1 = \frac{1}{3}, \phi_2 = \frac{2}{3}, \phi_3 = 1$, (Fig. 4) then

$$t_i = \tfrac{9}{2}\phi^3[t_i^3 - 3t_i^2 + 3t_i^1 - t_i^1)] + \tfrac{9}{2}\phi^2[-t_i^3 + 4t_i^2 - 5t_i^1 + 2t_i^0] + \phi[t_i^3 - \tfrac{9}{2}t_i^2 + 9t_i^1 - \tfrac{11}{2}t_i^0] + t_i^0$$

which results finally in a set of 3 equations depending on the choice of integration limits: for $\langle 0, 1\rangle$, $\langle 0, \frac{2}{3}\rangle$, $\langle \frac{1}{3}, 1\rangle$ the equations are symmetric:

$$\sum_k \bar{a}_{ik} \tfrac{1}{8} [(t_i^3 + 3t_i^2 + 3t_i^1 + t_i^0) - (t_k^3 + 3t_k^2 + 3t_k^1 + t_k^0)] + t_i^3 - t_i^0 = 0 \quad i = 1, \ldots, n \qquad (8)$$

$$\sum_k \bar{a}_{ik} \tfrac{1}{9} [(t_i^2 + 4t_i^1 + t_i^0) - (t_k^2 + 4t_k^1 + t_k^0)] + t_i^2 - t_i^0 = 0 \quad i = 1, \ldots, n \qquad (9)$$

$$\sum_k \bar{a}_{ik} \tfrac{1}{9} [(t_i^3 + 4t_i^2 + t_i^1) - (t_k^3 + 4t_k^2 + t_k^1)] + t_i^3 - t_i^1 = 0 \quad i = 1, \ldots, n. \qquad (10)$$

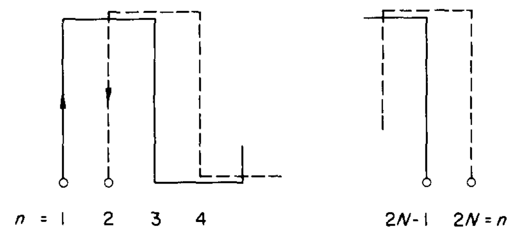

FIG. 5a. Flow path for parallel-flow exchanger.

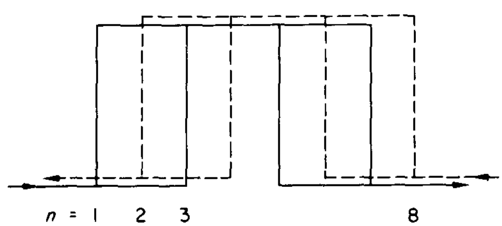

FIG. 5b. Flow path for mixed-flow exchanger.

In order to assess the accuracy of the present solutions, numerical studies of some of the problems solved analytically by Mennicke were performed using both second and third order polynomial approximation. Two types of exchangers analysed are shown in Fig. 5. Assuming

U is constant throughout the exchanger and in case 5b the flow rate in parallel branches is the same, heat transfer is described by two parameters $\bar{a}_{12} = N_{TUc}$ and $u = C_h/C_c$. In order to compare our results with earlier work [3], we introduce concept of effective heat transfer conductance U_e, which is defined by the equation

$$Q = C_c\,\Delta t_c = C_h\,\Delta t_h = (n-1)\,F U_e\,\Delta t_{\log} \tag{11}$$

Here, $\Delta t_{\log}$ is the conventional logarithmic mean temperature difference. U_e is the heat transfer conductance, which would yield the same outlet temperatures in a countercurrent exchanger with the same rates and total area. Results of analytical solutions in (3) and (4) are then expressed in terms of the ratio

$$\frac{U_e}{U} = \frac{1}{n-1}\,\frac{1}{N_{TUc}}\,\frac{\Delta t_c}{\Delta t_{\log}}. \tag{12}$$

In more customary terms, U_e/U is the ratio of N_{TU}'s for countercurrent and given exchanger with the same effectiveness. Here the ratio U_e/U is essentially identical with the factor F_e as described by Kays and London [8]. The following tables show the comparison of the exact values, [3] and [4], and those calculated here for various combinations of parameters:

I. *Polynomials of second order*

$n = 4$, $u = 1$, exchanger 5a

N_{TUc}	0·8	0·7	0·6	0·5	0·4	0·3	0·2
U_e/U exact	0·6367	0·6907	0·7482	0·8075	0·8657	0·9188	0·9618
U_e/U calc	0·6294	0·6853	0·7445	0·8053	0·8646	0·9183	0·9617
Error %	−1·15	−0·78	−0·5	−0·27	−0·12	−0·05	−0·01

$n = 10$, $u = 1$, exchanger 5a

N_{TUc}	0·8	0·7	0·6	0·5	0·4	0·3	0·2
U_e/U exact	0·5406	0·6014	0·6689	0·7413	0·8153	0·8857	0·9455
U_e/U calc	0·5330	0·5957	0·6649	0·7389	0·8144	0·8857	0·9447
Error %	−1·41	−0·95	−0·76	−0·33	−0·11	+0·0	−0·09

$n = 8$, $u = 1$, exchanger 5b

N_{TUc}	0·8	0·7	0·6	0·5	0·4	0·3	0·2
U_e/U exact	0·7599	0·7969	0·8343	0·8713	0·9069	0·9401	0·9669
U_e/U calc	0·7579	0·7951	0·8331	0·8706	0·9063	0·9390	0·9668
Error %	−0·33	−0·23	−0·14	−0·08	−0·06	−0·11	−0·01

II. *Polynomials of third order*

$n = 4$, $u = 1$, exchanger 5a

N_{TUc}	0·8	0·7	0·6	0·5	0·4	0·3	0·2
U_e/U exact	0·6367	0·6907	0·7482	0·8075	0·8657	0·9188	0·9618
U_e/U calc	0·6376	0·6913	0·7486	0·8078	0·8658	0·9188	0·9618
Error %	0·14	0·1	0·05	0·03	0·01	0·00	0·00

$n = 10, u = 1$, exchanger 5a

N_{TUc}	0·8	0·7	0·6	0·5	0·4	0·3	0·2
U_e/U exact	0·5406	0·6014	0·6689	0·7413	0·8153	0·8857	0·9455
U_e/U calc	0·5416	0·6022	0·6694	0·7417	0·8158	0·8863	0·9456
Error %	0·185	0·13	0·07	0·05	0·06	0·07	0·01

$n = 4, u = 5·0$, exchanger 5a

N_{TUc}	0·8	0·7	0·6	0·5	0·4	0·3	0·2
U_e/U exact	0·8734	0·9022	0·9278	0·9498	0·9679	0·9820	0·9920
U_e/U calc	0·8740	0·9025	0·9279	0·9498	0·9679	0·9819	0·9956
Error %	0·07	0·03	0·01	0·00	0·00	0·01	0·36

It is obvious that polynomials of the third order yield practically exact solutions for the stated range of variables; these ranges were of practical use for design (for $n = 10$, $N_{TUc} = 0·8$, $u = 1$ heating of fluids is about to 80 per cent of the inlet temperature differences of the fluids). The computer time involved is almost entirely spent in the solution of the matrix equations of the order $3n$ and $4n$ for polynomials of second and third order respectively. On the IBM 360/50 one run with $n = 10$, with a third order polynomial requires about 7 s (employing a standard subroutine for matrix inversion).

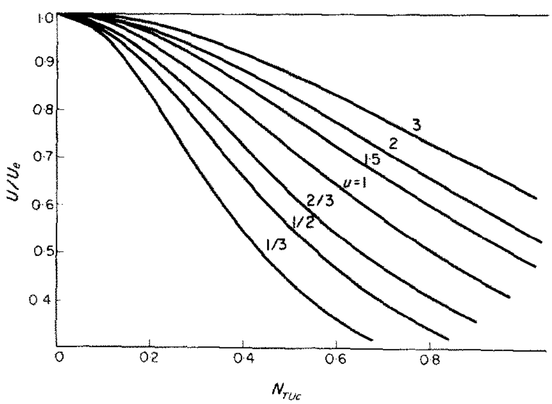

FIG. 7. Correlation chart for exchanger 5a: $n = 30$, third order polynomials.

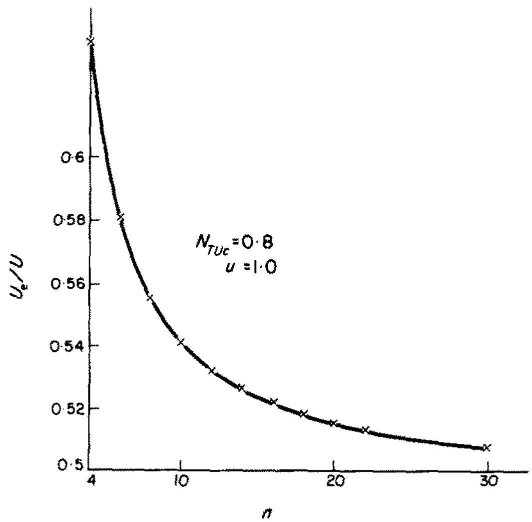

FIG. 6. Dependence of computed U_e/U on the number of channels n for exchanger 5a.

As a practical application, universal correlation charts for design of exchangers according to Fig. 5 can be developed. Figure 6 shows U_e/U as a computed function of the number of channels for $N_{TUc} = 0·8$ and $u = 1$. It can be seen, that for a large number of channels, the value of U_e/U approaches some limiting value. Therefore, from calculating charts $U_e/U = f(N_{TUc}, U)$ for two large values of n, interpolation or extrapolation will yield a value of U_e/U for any set of parameters, N_{TUc}, u, n with sufficient accuracy. As an example, such a chart is given for exchanger 5a and $n = 30$ in Fig. 7.

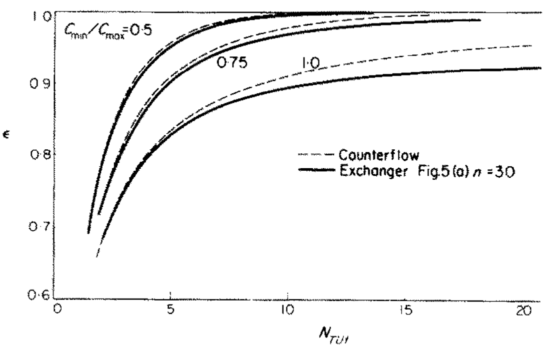

FIG. 8. $N_{TU_t} - \varepsilon$ relationship, exchanger 5a: $n = 30$, third order polynomials.

A similar approach will also probably be possible for different configurations, as suggested by the existence of asymptotic solutions for large n in [6].

These data can be converted in order to express our results in more customarily terms of effectiveness. Figure 8 shows $N_{TU_t} - \varepsilon$ relation for the same case as given in Fig. 7. N_{TU_t} in this figure is the total N_{TU} of the exchanger, i.e. $N_{TU_t} = (n-1)\ FU/C_{min}$. Dashed lines show corresponding effectiveness curves for the counter-current case (see [8]). We can see that the effectiveness of counter-current exchanger is always higher, as expected. This is also expressed by the fact that $U_e/U < 1$, i.e. we always need less N_{TU} for the ideal case of a counter-current exchanger compared to the actual multipass exchanger, in order to achieve the same rate of heat exchange. Effectiveness of the exchanger improves with the number of channels, as can be seen from Fig. 9 which shows an example of the effectiveness of the exchanger according to Fig. (5a) for different numbers of channels at fixed $C_{min}/C_{max} = 1$.

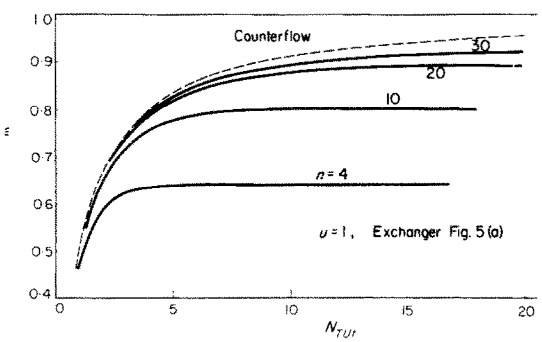

FIG. 9. Exchanger effectiveness vs. number of channels, exchanger 5a.

EXTENSION OF THE METHOD

The method as developed can also, with slight modification, be utilized to consider the nonlinear and transient problems. Further, different basis-functions can be used to approximate stream temperature.

a. *The nonlinear problem*

(1) Change of area with distance

(2) Change of U, C with temperature, both of which can be described again by equation (2), where the coefficients are now functions of temperature and ϕ,

$$\frac{dt_i}{d\phi} + \sum_{k=1}^{n} \bar{a}_{ik}(t_i, t_k, \phi)(t_i - t_k) = 0 \qquad i = 1, \ldots, n$$

where

$$\bar{a}_{ik} = \frac{LU(t_i, t_k)\,p(\phi)}{c_i(t_i)}.$$

Assuming that the heat transfer coefficients on both sides of the wall between channels i and k are independent,

$$U(t_i, t_k) = \frac{1}{1/h_i(t_i) + 1/h_k(t_k) + R_w}$$

where R_w is the resistance on the wall. The case of heat transfer coefficients being interdependent, that is h_k being a function of t_k and t_i can also be considered [10]. The basic idea for using the method for this case is to take integrals from approximating function between the next nodal points, that is

$$\int_{\phi_0=0}^{\phi_1}, \int_{\phi_1}^{\phi_2}, \int_{\phi_2}^{\phi_3}, \ldots, \int_{\phi_{m}-1}^{\phi_m=1}.$$

In each of those subintegrals, the functions $\bar{a}_{ik}$ are approximated in some way using nodal values of temperatures and substituting in the basic integral equation (4). Only two of the simplest possibilities will be shown although, of course, the procedure can be extended to higher order approximation of coefficients.

(i) *Approximation by piecewise constant function.* In this case define

$$\bar{a}_{ik}^{j} = \bar{a}_{ik}(\bar{t}_i^j, \bar{t}_k^j, \phi_j), \quad j = 0, 1, \ldots, m$$

with the final set of equations being in the form

$$\sum_k \frac{\bar{a}_{ik}^{j} + \bar{a}_{ik}^{j+1}}{2} \int_{\phi_j}^{\phi_{j+1}} (\bar{t}_i - \bar{t}_k)\, d\phi + \bar{t}_i^{j+1} - t_i^j = 0 \quad \begin{matrix} j = 0, \ldots, m-1 \\ i = 1, \ldots, n. \end{matrix} \tag{13}$$

These equations are nonlinear, as $\bar{a}_{ik}^{j}$ depends on the solution, however, they can be handled by standard techniques of numerical analysis [11].

(ii) *Linear approximation in* $\langle \phi_j, \phi_j + 1 \rangle$. If

$$\hat{a}_{ik} = \bar{a}_{ik}^{j} + \frac{\bar{a}_{ik}^{j+1} - \bar{a}_{ik}^{j}}{\phi_{j+1} - \phi_j}(\phi - \phi_j) = \bar{a}_{ik}^{j} + R^{j,j+1}(\phi - \phi_j),$$

linear approximation for $\phi \in \langle \phi_j, \phi_{j+1} \rangle$ is defined. Substitution in (4) yields

$$\sum_k \bar{a}_{ik}^{j} \int_{\phi_j}^{\phi_{j+1}} (t_i - t_k)\, d\phi + \sum_k R^{j,j+1} \int_{\phi_j}^{\phi_{j+1}} (\phi - \phi_j)(t_i - t_k)\, d\phi + t_i^{j+1} - t_i^j = 0 \quad \begin{matrix} j = 0, 1, \ldots, m-1 \\ i = 1, \ldots, n. \end{matrix} \tag{14}$$

The integrals in the second sum are after integration against linear functions of nodal temperatures. Thus, only the coefficients in the equations will be different from (i). This procedure can be continued by taking $\bar{a}_{ik}$ as a polynomial of second order etcetera. It can be proved as in the linear case that if this method converges with $m \to \infty$ to some differentiable function then it converges to the true solution.

(b) *The transient response of the exchanger*

The method can also be used for solution of the transient problem. The mathematical models of the transient problem are surveyed, for example, in [12]. However, if attempts are made to extend the present method for such a problem, use cannot be made of the model which considers also a coordinate across the flow. For the plug flow model, as defined in [12], integration of the differential equation for stream i (which is similar to equation (2) from [12]) across the stream, along with the application of Green's theorem, yields

$$\frac{\partial t_i}{\partial \phi} + \sum_{k=1}^{n} \bar{a}_{ik}(t_i - t_k) = -\frac{\partial t_i}{\partial \bar{\tau}_i}, \quad i = 1, \ldots, n \tag{15}$$

where $\bar{\tau}_i = \tau u_i/L$ is dimensionless time, u_i is volumetric velocity of stream i, that is $u_i = Q_i/F_{ci}$ where F_{ci} is the cross section of channel i.

Equations (15) differ from the basic equations (2) only by the addition of the time derivative term on the right hand side. Clearly, the same method can be applied provided the right hand side of the equation is known. Taking the approximation profile again as

$$\bar{t}_i = \sum_{k=0}^{m} c_{ik}\, \phi^k$$

but considering $c_{ik} = c_{ik}(\tau)$, results in a system of ordinary differential equations with respect to time for the coefficients c_{ik}. (This approach is similar to the so-called semi-descretization, used with finite differences for transient problems. see [13]). Using some finite different approximation for the time derivative, a recurrence formula, involving solution of a matrix problem, similar to that in the steady case, on each time step is obtained. It is obvious that all remarks about the extension to the nonlinear case also remain valid for the transient nonlinear solution.

(c) *Different basis functions*

From the point of view of numerical analysis, the choice of the polynomial approximation is only one of a number of possibilities of these so-called basis functions.

For instance, the simplest possibility is to make arbitrary sub-division of the interval (0, 1) on the ϕ axis and define $\bar{t}_i$ as the piecewise linear function between nodal values $\bar{t}_i^l$ and $\bar{t}_i^{l+1}$. Subintervals containing several points can also be joined on, use being made of a polynomial approximation as described before. In both these cases, the first derivative will not be continuous at joining points.

If smooth basis functions are required, Hermite interpolation functions or spline functions, which can be considered as generalized polynomials [14] may be chosen. In all these cases, the basis functions will yield a linear matrix problem for coefficients, and solutions may be obtained as described.

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SOLUTION APPROCHÉE D'ÉQUATIONS RELATIVES A DES ÉCHANGEURS DE CHALEUR MULTICANAUX POUR DES ÉCOULEMENTS PARALLÈLES ET AVEC MÉLANGE

Résume—Des solutions approchés pour la formulation linéaire globale de l'état permanent du transfert thermique dans des échangeurs de chaleur multi-canaux parallèles et à écoulement mélés sont obtenues par utilisation d'approximations polynomiales pour les températures dans chaque courant. Il en résulte des polynômes simples du second et troisème degré dans une marge d'erreur acceptable comparés avec les solutions exactes. La méthode est spécialement valable pour le traitement pratique d'échangeurs avec un grand nombre de canaux quand d'autres méthodes cessent d'être applicables. L'extension de la méthode au traitment du problème non-linéaire transitoire est esquissée.

NÄHERUNGSMETHODEN FÜR DIE LÖSUNG DER GLEICHUNGEN FÜR DEN PARALLEL- UND KREUZSTROM-WÄRMEÜBERTRAGER

Zusammenfassung—Näherungslösungen für die stationäre und vereinfachte lineare Formulierung der Wärmeübertragung in Vielkanal-Wärmeübertragern erhält man bei Verwendung von Approximations-Polynomen für die Temperatur in jedem Strom. Schon einfache Polynome vom zweiten und dritten Grad ergeben beim Vergleich mit der exakten Lösung einen tolerierbaren Fehler. Die Methode ist besonders geeignet für die praktische Berechnung des Wärmeübertragers mit einer grossen Zahl von Kanälen, bei denen andere Methoden versagen. Eine Erweiterung der Methode für die Behandlung nichtlinearer und instationärer Probleme wird dargestellt.

ПРИБЛИЖЕННЫЙ МЕТОД РЕШЕНИЯ УРАВНЕНИЙ ДЛЯ МНОГОКАНАЛЬНЫХ ТЕПЛООБМЕННИКОВ С ПАРАЛЛЕЛЬНЫМИ И СМЕШАННЫМИ ТЕЧЕНИЯМИ

Аннотация—С помощью полиномных приближений для температур в каждом течении получены приближенные решения для стационарной приведенной линейной формулировки теплообмена в многоканальном теплообменнике с параллельными и смешанными течениями. Полиномы второй и третьей степеней дают допустимые погрешности по сравнению с точными решениями. Метод особенно подходит для практических расчетов теплообменников, имеющих большое число каналов, когда другие методы неприменимы. Намечено обобщение метода для нелинейных и нестационарных задач.